

Three Results on Making Change (An Exposition)

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Abstract

Let a_1, \dots, a_L be relatively prime. We think of them as coin denominations. Let $M = LCM(a_1, \dots, a_L)$ and let $CH(n)$ be the number of ways to make change of n cents. We show there is an *exact* piece wise formula for $CH(n)$. The pieces are polynomials that depend on $n \bmod M$. We show that many of the pieces agree on all but the constant term. These results are not new; however, our treatment is self-contained, unified, and elementary.

1 Introduction

Throughout this paper we let:

1. a_1, a_2, \dots, a_L be coin denominations. Assume you have an unlimited number of each coin. They need not be distinct. Think of having red nickels and blue nickels.
2. $M = LCM(a_1, \dots, a_L)$.
3. $M' = LCM(GCD(a_1, a_2), GCD(a_1, a_3), \dots, GCD(a_{L-1}, a_L))$.

Notation 1.1 If a_1, \dots, a_L are given then $CH(n)$ is the number of ways to make change of n cents. Sylvester called $CH(n)$ *the denumerant*.

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Determining $CH(n)$ is known as *the problem of finding the coefficients of the Sylvester denominator*. It is related to the well known Frobenius problem: *What is the largest n such that $CH(n) = 0$?* Modern papers on this topic tend to use advanced mathematics. We list some of the papers [1, 3, 4, 5, 7, 10, 11, 13] and some of the books [2, 6, 8, 12] where the problem is discussed.

We obtain an *exact* piece wise formula for $CH(n)$ and then refine it. Our results are not new; however, our treatment is self-contained, unified, and elementary treatment. We include the polynomials for several coin sets in the Appendix and make some observations and conjectures.

Our results begin with the following premise: $\{a_1, \dots, a_L\}$ is a set of coin denominations that are relatively prime with M, M' as above. Note that if the coin set is $\{1, 5, 10, 25\}$ then $M = 50$ and $M' = 5$. This is typical in that M' is usually much less than M .

Our first result is that there exist $h_0, h_1, \dots, h_{M-1} \in \mathbb{Q}[x]$ of degree $L - 1$ such that

$$CH(n) = h_{n \bmod M}(n).$$

Bell [7] attributes this result to Sylvester and Cayley and refers the reader to Dickson [9] (vol 2) for the history of denumerants up to 1919. Bell [7] gave a proof that is simpler than the proof of Sylvester and Cayley. Our proof is similar to Bell's.

Our second results shows that if you ignore the constant term then many of the polynomials are identical. Keep in mind that M' is usually much less than M . We show that there there exist $h'_0, \dots, h'_{M'-1} \in \mathbb{Q}[x]$ of degree $L - 1$ and rationals b_0, \dots, b_{M-1} such that

$$CH(n) = h'_{n \bmod M'}(n) + b_{n \bmod M}.$$

This can be derived from Theorem 1.7 (page 15) of the book by Beck and Robins [6] and probably from other formulas for $CH(n)$ as well. Our proof is simpler than theirs and may be new.

Our third result is that

$$CH(n) = \frac{n^{L-1}}{(L-1)!a_1a_2 \cdots a_L} + O(n^{L-2}).$$

This result is attributed to Schur by Riordan [12], Wilf [14], and all of the papers and books cited above that mention it. Our proof is similar to the one in Wilf's book on generating functions [14]. After we prove this we will give a geometric interpretation.

We then obtain, as a corollary, three theorems that are similar to those stated above; however, they apply to *any* coin set $\{a_1, \dots, a_L\}$.

2 Needed Lemmas

We obtain the Taylor expansion for $\frac{1}{(1-x)^L}$ via combinatorics, not calculus.

Lemma 2.1 *For all L , $\frac{1}{(1-x)^L} = \sum_{n=0}^{\infty} \binom{L-1+n}{L-1} x^n$.*

Proof: We rewrite this as

$$(1 + x + x^2 + \cdots)^L = \sum_{n=0}^{\infty} \binom{L-1+n}{L-1} x^n.$$

Let $S(L, n)$ be the number of solutions of $x_1 + \cdots + x_L = n$ where $x_i \geq 0$. Clearly the coefficient of x^n of the LHS is $S(L, n)$. By viewing $S(L, n)$ as the number of ways of permuting n dots and $L-1$ bars we see that $S(L, n) = \binom{L-1+n}{L-1}$. Hence the LHS and the RHS are the same.

■

We leave the following lemma to the reader.

Lemma 2.2 *If $\zeta^a = 1$ then there exists d such that ζ is a primitive d th root of unity and d divides a .*

Lemma 2.3 *Let a_1, \dots, a_L be relatively prime. Let $g(x) = (x^{a_1} - 1) \cdots (x^{a_L} - 1)$. When $g(x)$ is factored completely into linear terms the factor $(x - 1)$ occurs L times and all of the other linear factors occur $\leq L - 1$ times.*

Proof: Let ζ be a root of $g(x)$. We are concerned with the multiplicity of ζ . By Lemma 2.2 ζ is a primitive d th root of unity where d divides some a_i . We denote this d by d_ζ . The multiplicity of ζ is $|\{1 \leq j \leq L : d_\zeta | a_j\}|$. Since the a_i 's are relatively prime the only ζ with $|\{1 \leq j \leq L : d_\zeta | a_j\}| = L$ is $\zeta = 1$. ■

Lemma 2.4 *Let a_1, a_2 be integers and ζ be a complex number. If $\zeta^{a_1} = 1$ and $\zeta^{a_2} = 1$ then $\zeta^{GCD(a_1, a_2)} = 1$.*

Proof: By Lemma 2.2 ζ is a primitive d th root of unity where d divides a_1 and a_2 . Clearly d divide $GCD(a_1, a_2)$. Hence $\zeta^{GCD(a_1, a_2)} = 1$. ■

Lemma 2.5 *Let f be a polynomial of degree $L - 1$. If there are L rationals r such that $f(r)$ is rational then all of the coefficients of f are rational.*

Proof: Assume r_1, \dots, r_L are rational and $f(r_1), \dots, f(r_L)$ are rational.

Let $h_j(x) = \prod_{i=1, i \neq j}^L \frac{x - r_i}{r_j - r_i}$. Note that (1) for all $x \in \{r_1, \dots, r_L\} - \{r_j\}$, $h_j(x) = 0$, (2) $h_j(r_j) = 1$, and (3) h_j is a polynomial over the rationals of degree $L - 1$.

Let $F(x) = \sum_{j=1}^L f(r_j) h_j(x)$. Clearly, for all $1 \leq i \leq L$, $F(r_i) = f(r_i)$. Hence F and f are polynomials of degree $L - 1$ that agree on L points, so $f = F$. Since F has rational coefficients, f has rational coefficients. ■

Note 2.6 The above proof is based on a well-known technique, called Lagrange interpolation, to find a polynomial that goes through a given set of points.

3 Main Theorem

Theorem 3.1 *Let $a_1, \dots, a_L \in \mathbb{N}$ be relatively prime. Let $M = \text{LCM}(a_1, \dots, a_L)$ and $M' = \text{LCM}(\text{GCD}(a_1, a_2), \text{GCD}(a_1, a_3), \dots, \text{GCD}(a_{L-1}, a_L))$.*

1. *There exists $h_0, h_1, \dots, h_{M-1} \in \mathbb{Q}[x]$ of degree $L - 1$ such that $CH(n) = h_{n \bmod M}(n)$.*
2. *There exists $h'_0, \dots, h'_{M'-1} \in \mathbb{Q}[x]$ of degree $L - 1$, and rationals b_0, \dots, b_{M-1} such that*

$$CH(n) = h'_{n \bmod M'}(n) + b_{n \bmod M}.$$

3.

$$CH(n) = \frac{n^{L-1}}{(L-1)!a_1a_2 \cdots a_L} + O(n^{L-2}).$$

Proof:

The value of $CH(n)$ is the coefficient of x^n in

$$\begin{aligned} f(x) &= (1 + x^{a_1} + x^{2a_1} + \cdots)(1 + x^{a_2} + x^{2a_2} + \cdots) \cdots (1 + x^{a_L} + x^{2a_L} + \cdots) \\ &= \frac{1}{(1-x^{a_1})(1-x^{a_2}) \cdots (1-x^{a_L})}. \end{aligned}$$

Assume $a_1 \leq \cdots \leq a_L$ and i_o is such that $a_{i_o} \geq 2$. (If no such i_o exists then $(\forall n)[CH(n) = 1]$ and our theorem is trivially true.) For all $i_o \leq i \leq L$, $1 \leq j \leq a_i - 1$, let α_{ij} be the j th a_i th root of unity (we think of 1 as being the 0th root of unity). Let n_{ij} be the number of times the factor $(1 - \alpha_{ij}x)$ appears in $(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_L})$. Since $a_{i_o} \geq 2$ none of the α_{ij} are 1. This will be important in the proof of part 3.

We rewrite $f(x)$ using partial fractions and Lemma 2.1 to obtain

$$\begin{aligned} f(x) &= \frac{1}{(1-x)^L \prod_{i=i_o}^L \prod_{j=1}^{a_i-1} (1 - \alpha_{ij}x)^{n_{ij}}} = \sum_{i=i_o}^L \frac{A_i}{(1-x)^i} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{ij}} \frac{A_{ijk}}{(1 - \alpha_{ij}x)^k} \\ &= \sum_{i=i_o}^L \sum_{n=0}^{\infty} A_i \binom{n+i-1}{i-1} x^n + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{ij}} \sum_{n=0}^{\infty} A_{ijk} \binom{n+k-1}{k-1} \alpha_{ij}^n x^n \end{aligned}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=i_o}^L A_i \binom{n+i-1}{i-1} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{ij}} A_{ijk} \binom{n+k-1}{k-1} \alpha_{ij}^n \right) x^n.$$

Hence

$$CH(n) = \sum_{i=i_o}^L A_i \binom{n+i-1}{i-1} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{ij}} A_{ijk} \binom{n+k-1}{k-1} \alpha_{ij}^n.$$

By Lemma 2.3 $n_{ij} \leq L-1$. Hence we can write $CH(n)$ as $\sum_{e=0}^{L-1} COE(n, e) n^e$ where the $COE(n, e)$ are functions of the α_{ij}^n .

1) Since α_{ij} is an a_i th root of unity, $\alpha_{ij}^n = \alpha_{ij}^{n \bmod M}$. Hence, for all $0 \leq e \leq L-1$, $COE(n, e) = COE(n \bmod M, e)$. Therefore the coefficients only depend on $n \bmod M$. For $0 \leq r \leq M-1$ let

$$h_r(n) = \sum_{i=i_o}^L A_i \binom{n+i-1}{i-1} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{ij}} A_{ijk} \binom{n+k-1}{k-1} \alpha_{ij}^r = \sum_{e=0}^{L-1} COE(r, e) n^e.$$

Clearly h_r is a polynomial in n of degree $L-1$ and $CH(n) = h_{n \bmod M}(n)$. Since there is an infinite number of $n \in \mathbb{N}$ (namely all $n \equiv r \pmod{M}$) such that $h_r(n) \in \mathbb{N}$, by Lemma 2.5 the coefficients of h_r are rational numbers. Hence $h_r(x) \in \mathbb{Q}[x]$.

2) For $0 \leq r \leq M-1$ let

$$h'_r(n) = \sum_{i=i_o}^L A_i \binom{n+i-1}{i-1} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=2}^{n_{ij}} A_{ijk} \binom{n+k-1}{k-1} \alpha_{ij}^r.$$

Note that $h_r(n)$ and $h'_r(n)$ only differ with regard to whether k starts at 1 or 2. For $0 \leq r \leq M-1$ let

$$b_r = h_r(n) - h'_r(n) = \sum_{i=i_o}^L \sum_{j=1}^L A_{ij1} \alpha_{ij}^r.$$

Clearly the b_r 's are constants (we later show they are rational) and

$$CH(n) = h_{n \bmod M}(n) = h'_{n \bmod M'}(n) + b_{n \bmod M}.$$

For $e \geq 1$, the coefficient of n^e in both $h_r(n)$ and $h'_r(n)$ are the same. We need to show that, for $e \geq 1$, $COE(n, e) = COE(n \bmod M', e)$. Let $e \geq 1$. Let X_{ke} be such that $\binom{n+k-1}{k-1} = \sum_{e=0}^{k-1} X_{ke} n^e$. Then

$$COE(n, e) = \sum_{i=i_o}^L A_i X_{ie} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=2}^{n_{ij}} A_{ijk} X_{ke} \alpha_{ij}^n$$

Fix i, j . If $n_{ij} \leq 1$ there is no k with $2 \leq k \leq n_{ij}$; therefore we assume $n_{ij} \geq 2$. So the term $(1 - \alpha_{ij} x)$ appears at least twice when factoring $(1 - x^{a_1}) \cdots (1 - x^{a_L})$. Therefore there exists $i' \neq i$ such that α_{ij} is an $a_{i'}$ th root of unity. Since α_{ij} is also an a_i th root of unity, by Lemma 2.4, α_{ij} is a d th root of unity where $d = GCD(a_i, a_{i'})$. Since d divides M' , $\alpha_{ij}^{M'} = 1$, hence $\alpha_{ij}^n = \alpha_{ij}^{n \bmod M'}$.

Therefore

$$COE(n, e) = \sum_{i=i_o}^L A_i X_{ie} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=2}^{n_{ij}} A_{ijk} X_{ke} \alpha_{ij}^{n \bmod M'}$$

which clearly only depends on $n \bmod M'$.

Fix $0 \leq r \leq M' - 1$ and $0 \leq s \leq M - 1$ such that there is an infinite number of $n \in \mathbb{N}$ with $n \equiv r \pmod{M'}$ and $n \equiv s \pmod{M}$. Hence, for an infinite number of $n \in \mathbb{N}$, $h'_r(n) + b_s = CH(n) \in \mathbb{N}$. By Lemma 2.5 $h'_r(x) \in \mathbb{Q}[x]$ and the b_s 's are rationals.

3)

$$CH(n) = \sum_{i=i_o}^L A_i \binom{n+i-1}{i-1} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{ij}} A_{ijk} \binom{n+k-1}{k-1} \alpha_{ij}^n = A_L \binom{n+L-1}{L-1}.$$

We find A_L .

$$\frac{1}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_L})} = \sum_{i=i_o}^L \frac{A_i}{(1-x)^i} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{ij}} \frac{A_{ijk}}{(1-\alpha_{ij}x)^k}.$$

Multiply both sides by $(1-x)^L$ to get

$$\frac{(1-x)^L}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_L})} = A_L + \sum_{i=i_o}^{L-1} A_i(1-x)^{L-i} + \sum_{i=i_o}^L \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{ij}} \frac{A_{ijk}(1-x)^L}{(1-\alpha_{ij}x)^k}.$$

The left hand side can be rewritten as

$$\frac{1}{(1+x+x^2+\cdots+x^{a_1-1})(1+x+x^2+\cdots+x^{a_2-1})\cdots(1+x+x^2+\cdots+x^{a_L-1})}.$$

As x approaches 1 (from the left), the LHS approaches $\frac{1}{a_1 a_2 \cdots a_L}$. Since for all i, j , $\alpha_{ij} \neq 1$, as x approaches 1, the RHS approaches A_L . Hence $A_L = \frac{1}{a_1 a_2 \cdots a_L}$ and $COE(n, L-1) = \frac{1}{(L-1)! a_1 a_2 \cdots a_L}$.

■

An equivalent definition of $CH(n)$ is the number of integer points in the set

$$P_n = \{(x_1, \dots, x_L) : \text{all } x_i \geq 0 \text{ and } \sum_{i=1}^L a_i x_i = n\}.$$

The quantity $\frac{1}{(L-1)! a_1 a_2 \cdots a_L}$ is the volume of P_1 . Hence Theorem 3.1.3 says that the number of integer points in P_n is approximately $VOL(P_1)n^{L-1}$. Counting the number of integer points in a convex polytope, including the application to coin problems, is studied by Beck and Robins [6].

The following is an easy corollary of Theorem 3.1.

Corollary 3.2 *Let a_1, \dots, a_L have greatest common divisor d . Let $M = LCM(a_1/d, \dots, a_L/d)$ and $M' = LCM(GCD(a_1/d, a_2/d), GCD(a_1/d, a_3/d), \dots, GCD(a_{L-1}/d, a_L/d))$.*

1. *If $n \not\equiv 0 \pmod{d}$ then $CH(n) = 0$.*

2. *There exists $h_0, h_1, \dots, h_{M-1} \in \mathbb{Q}[x]$ of degree $L - 1$ such that if $n \equiv 0 \pmod{d}$ then $CH(n) = h_{n \bmod M}(n)$.*
3. *There exists $h'_0, \dots, h'_{M'-1} \in \mathbb{Q}[x]$ of degree $L - 1$, and rationals $b_0, \dots, b_{M-1} \in \mathbb{Q}$, such that if $n \equiv 0 \pmod{d}$ then $CH(n) = h'_{n \bmod M'}(n) + b_{n \bmod M}$.*
4. *If $CH(n)$ is restricted to $n \equiv 0 \pmod{d}$ then*

$$CH(n) = \frac{n^{L-1}d^L}{(L-1)!a_1a_2 \cdots a_L} + O(n^{L-2}).$$

4 Examples and Conjectures

In the Appendices we present, for a variety of coin sets, M , M' , $h_{0 \leq r \leq M}$, $h'_{0 \leq r \leq M'}$, and upper/lower bounds on the b_i 's. When calculating M' we omit the pairs of the form $GCD(1, a_j)$ since $GCD(1, a_i) = 1$. For h'_r we take the version with 0 constant term. We obtained the polynomials via Lagrange interpolation. In this section we describe the results and what they might mean.

Let the coin set be $\{1, 5, 10, 25\}$, so that $M = 50$ and $M' = 5$. In Appendix A we have the polynomials $h_{0 \leq r \leq 49}$. Note that (1) if $r_1 \equiv r_2 \pmod{5}$ then h_{r_1} and h_{r_2} agree on all the coefficients except the constant term, and (2) all of the leading coefficients are the same. This is predicted by Theorem 3.1. Also note that (1) all of the coefficients are positive, (2) for all coefficients c , $2(L-1)a_1 \cdots a_L c \in \mathbb{N}$, and (3) the b_i 's are small. Do (1), (2), (3) hold for all coin sets?

4.1 Are the Coefficients Always Positive?

We refer to the statement

for all coin sets all of the coefficients of the h -polynomials associated to them are positive
as (1).

Clearly (1) does not always hold: if a coin set has $a_1 \neq 1$ then $CH(1) = 0$ so some coefficient of h_1 has to be negative. In Appendices C and E we present the polynomials for the coin sets $\{2, 3, 4\}$ and $\{3, 5, 6\}$. For $\{2, 3, 4\}$ three of the polynomials have a negative constant term. For $\{3, 5, 6\}$ eleven of the polynomials have a negative constant term. All of the non-constant terms have positive coefficients.

Does (1) hold if $a_1 = 1$? Alas no. Of the 138 polynomials for the coin set $\{1, 4, 6, 11\}$, three of them have a negative constant term. We present these three polynomials in Appendix G. For all of the polynomials, all of the non-constant terms have positive coefficients.

Does (1) hold if we only look at the non-constant terms? If we allow a coin denomination to appear twice then no. In Appendix I we present the polynomials for the coin set $\{1, 19, 19, 20\}$ that have negative linear term. Of the 380 total polynomials there are 60 (or $3/19$) that have a negative linear term. We also have the following empirical results, which we do not give the polynomials for: $1/7$ of the polynomials for $(1, 21, 21, 22)$ have a negative linear term.

Based on our empirical evidence and talking to Matthias Beck and Michelle Vergne (experts in the field) we have the following conjectures.

1. If a_1, a_2, \dots, a_L are relatively prime then all of the associated polynomials have positive coefficients except possibly the constant term. (It might be easier to prove the $a_1 = 1$ case.)
2. If a_1, a_2, \dots, a_L are relatively prime and $a_1 = 1$ then all of the associated polynomials have positive coefficients.
3. (Michelle Vergne emailed us this conjecture) For x large and $x < y$ some of the associated polynomials to $(1, x, x, y)$ will have a negative linear term.

4.2 Is $2(L-1)a_1 \cdots a_L c$ Always an Integer?

We refer to the statement

for the coin set $\{a_1, \dots, a_L\}$, for all coefficients c of the h_r 's, $2(L-1)a_1 \cdots a_L c \in \mathbb{Z}$

as (2).

Statement (2) holds for all of the coin sets we have looked at. There is a known theorem which may be relevant here. We describe it.

A *convex rational polytope* is an intersection of halfspaces such that all of the corner points have rational coordinates. Recall that $CH(n)$ is the number of integer points in the convex rational polytope

$$P_n = \{(x_1, \dots, x_L) : \text{all } x_i \geq 0 \text{ and } \sum_{i=1}^L a_i x_i = n\}.$$

In Beck and Robins [6] Theorem 3.20 (page 80) states (roughly) that the number of integer points in a parameterized convex rational polytope is a piecewise polynomial. Their Exercise 3.33 (Page 87) states that for L -dimensional rational polytopes in \mathbb{R}^L , for all coefficients c of those polynomials, $L!c \in \mathbb{Z}$. Our P_n is not L -dimensional and hence their Exercise does not apply. It is plausible that their Exercise can be modified to hold for polytopes that are not L -dimensional, or polytopes that are exactly of the type of P_n above, to yield (2).

4.3 Are the b_i 's Small?

For the coin sets $\{1, 5, 10, 25\}$, $\{2, 3, 4\}$, $\{3, 5, 6\}$, and $\{1, 4, 6, 11\}$ the b_i 's are all in $[-0.4277, 1.3636]$. The smallest difference between the b_i 's is 1.0962 and the largest difference is 1.4277.

One conjecture is that there is some constant B such that for *all* coin sets the b_i 's are in $[-B, B]$. Another conjecture is that there is some slow growing function $h(L, a_1, \dots, a_L)$ such that for the coin set a_1, \dots, a_L all of the b_i 's are in $[-h(L, a_1, \dots, a_L), h(L, a_1, \dots, a_L)]$. Similar conjectures can be made for the difference.

All of the coin sets above have no repeated coins. For the coin set $\{1, 19, 19, 20\}$ the smallest b_i is -6.3644 and the largest b_i is 7.0953, for a difference of 13.4597. It may be that such coin sets behave very differently. Hence we only make the above conjectures for coin sets where all of the

coins are distinct.

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We would also like to thank the referees. They made comments that improved the paper considerably. In particular, the proofs in Sections 2 are much improved, and the proof of Theorem 3.1 is somewhat less cumbersome.

A h_r Polynomials for $\{1, 5, 10, 25\}$

$$M = LCM(1, 5, 10, 25) = 50.$$

$h_0(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + 1$	$h_5(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + \frac{7}{8}$
$h_1(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{4161}{5000}$	$h_6(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{442}{625}$
$h_2(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{426}{625}$	$h_7(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{2783}{5000}$
$h_3(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{2737}{5000}$	$h_8(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{264}{625}$
$h_4(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{268}{625}$	$h_9(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{1519}{5000}$
$h_{10}(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + \frac{6}{5}$	$h_{15}(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + \frac{7}{8}$
$h_{11}(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{5161}{5000}$	$h_{16}(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{442}{625}$
$h_{12}(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{551}{625}$	$h_{17}(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{2783}{5000}$
$h_{13}(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{3737}{5000}$	$h_{18}(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{264}{625}$
$h_{14}(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{393}{625}$	$h_{19}(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{1519}{5000}$
$h_{20}(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + \frac{4}{5}$	$h_{25}(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + \frac{7}{8}$
$h_{21}(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{3161}{5000}$	$h_{26}(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{442}{625}$
$h_{22}(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{301}{625}$	$h_{27}(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{2783}{5000}$
$h_{23}(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{1737}{5000}$	$h_{28}(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{264}{625}$
$h_{24}(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{143}{625}$	$h_{29}(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{1519}{5000}$
$h_{30}(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + 1$	$h_{35}(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + \frac{43}{40}$
$h_{31}(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{4161}{5000}$	$h_{36}(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{567}{625}$
$h_{32}(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{426}{625}$	$h_{37}(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{3783}{5000}$
$h_{33}(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{2737}{5000}$	$h_{38}(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{389}{625}$
$h_{34}(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{268}{625}$	$h_{39}(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{2519}{5000}$

$h_{40}(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + 1$	$h_{45}(x) = \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x + \frac{27}{40}$
$h_{41}(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{4161}{5000}$	$h_{46}(x) = \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x + \frac{317}{625}$
$h_{42}(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{426}{625}$	$h_{47}(x) = \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x + \frac{1783}{5000}$
$h_{43}(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{2737}{5000}$	$h_{48}(x) = \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x + \frac{139}{625}$
$h_{44}(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{268}{625}$	$h_{49}(x) = \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x + \frac{519}{5000}$

B h'_r **Polynomials for** $\{1, 5, 10, 25\}$

$$M' = LCM(GCD(5, 10), GCD(5, 25), GCD(10, 25)) = LCM(5, 5, 5) = 5.$$

$$\begin{aligned}
h'_0(x) &= \frac{1}{7500}x^3 + \frac{9}{1000}x^2 + \frac{53}{300}x \\
h'_1(x) &= \frac{1}{7500}x^3 + \frac{43}{5000}x^2 + \frac{1193}{7500}x \\
h'_2(x) &= \frac{1}{7500}x^3 + \frac{41}{5000}x^2 + \frac{1067}{7500}x \\
h'_3(x) &= \frac{1}{7500}x^3 + \frac{39}{5000}x^2 + \frac{947}{7500}x \\
h'_4(x) &= \frac{1}{7500}x^3 + \frac{37}{5000}x^2 + \frac{833}{7500}x
\end{aligned}$$

The smallest b_i is $\frac{519}{5000} = 0.1038$ and the largest b_i is $\frac{6}{5} = 1.2$. The difference between the largest and smallest is 1.0962.

C h_r **Polynomials for** $\{2, 3, 4\}$

$$M = LCM(2, 3, 4) = 12$$

$h_0(x) = \frac{1}{48}x^2 + \frac{1}{4}x + 1$	$h_4(x) = \frac{1}{48}x^2 + \frac{1}{8}x - \frac{7}{48}$
$h_1(x) = \frac{1}{48}x^2 + \frac{1}{8}x - \frac{7}{48}$	$h_5(x) = \frac{1}{48}x^2 + \frac{1}{4}x + \frac{3}{4}$
$h_2(x) = \frac{1}{48}x^2 + \frac{1}{4}x + \frac{5}{12}$	$h_6(x) = \frac{1}{48}x^2 + \frac{1}{8}x + \frac{5}{48}$
$h_3(x) = \frac{1}{48}x^2 + \frac{1}{8}x + \frac{7}{16}$	$h_7(x) = \frac{1}{48}x^2 + \frac{1}{4}x + \frac{2}{3}$

$$\begin{array}{lcl}
h_8(x) & = & \frac{1}{48}x^2 + \frac{1}{4}x + \frac{2}{3} \\
h_9(x) & = & \frac{1}{48}x^2 + \frac{1}{8}x + \frac{3}{16} \\
h_{10}(x) & = & \frac{1}{48}x^2 + \frac{1}{4}x + \frac{5}{12} \\
h_{11}(x) & = & \frac{1}{48}x^2 + \frac{1}{8}x + \frac{5}{48}
\end{array}
\left| \begin{array}{lcl}
h_{12}(x) & = & \frac{1}{48}x^2 + \frac{1}{8}x - \frac{7}{48} \\
h_{13}(x) & = & \frac{1}{48}x^2 + \frac{1}{4}x + \frac{5}{12} \\
h_{14}(x) & = & \frac{1}{48}x^2 + \frac{1}{8}x + \frac{7}{16} \\
h_{15}(x) & = & \frac{1}{48}x^2 + \frac{1}{4}x + \frac{2}{3}
\end{array} \right.$$

D h'_r **Polynomials for $\{2, 3, 4\}$**

$$M' = LCM(GCD(2, 3), GCD(2, 4), GCD(3, 4)) = LCM(1, 2, 1) = 2.$$

$$h'_0(x) = \frac{1}{48}x^2 + \frac{1}{4}x$$

$$h'_1(x) = \frac{1}{48}x^2 + \frac{1}{8}x$$

The smallest b_i is $-\frac{7}{48} = -0.1458$ and the largest b_i is 1. The difference between the largest and smallest is 1.1458.

E h_r **Polynomials for** $\{3, 5, 6\}$

$$M = LCM(3, 5, 6) = 30.$$

$h_0(x) = \frac{1}{180}x^2 + \frac{2}{15}x + 1$	$h_9(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{17}{36}$
$h_1(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{1}{36}$	$h_{10}(x) = \frac{1}{180}x^2 + \frac{2}{15}x + 1$
$h_2(x) = \frac{1}{180}x^2 + \frac{7}{90}x - \frac{8}{45}$	$h_{11}(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{77}{180}$
$h_3(x) = \frac{1}{180}x^2 + \frac{2}{15}x + \frac{11}{20}$	$h_{12}(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{1}{45}$
$h_4(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{8}{45}$	$h_{13}(x) = \frac{1}{180}x^2 + \frac{2}{15}x + \frac{7}{20}$
$h_5(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{17}{36}$	$h_{14}(x) = \frac{1}{180}x^2 + \frac{1}{45}x + \frac{2}{9}$
$h_6(x) = \frac{1}{180}x^2 + \frac{2}{15}x + 1$	$h_{15}(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{17}{36}$
$h_7(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{77}{180}$	$h_{16}(x) = \frac{1}{180}x^2 + \frac{2}{15}x + \frac{3}{5}$
$h_8(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{1}{45}$	$h_{17}(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{41}{180}$
$h_{18}(x) = \frac{1}{180}x^2 + \frac{2}{15}x + \frac{4}{5}$	$h_{27}(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{49}{180}$
$h_{19}(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{77}{180}$	$h_{28}(x) = \frac{1}{180}x^2 + \frac{2}{15}x + \frac{3}{5}$
$h_{20}(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{2}{9}$	$h_{29}(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{1}{36}$
$h_{21}(x) = \frac{1}{180}x^2 + \frac{2}{15}x + \frac{3}{4}$	$h_{30}(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{2}{9}$
$h_{22}(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{8}{45}$	$h_{31}(x) = \frac{1}{180}x^2 + \frac{2}{15}x + \frac{7}{20}$
$h_{23}(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{49}{180}$	$h_{32}(x) = \frac{1}{180}x^2 + \frac{1}{45}x + \frac{1}{45}$
$h_{24}(x) = \frac{1}{180}x^2 + \frac{2}{15}x + \frac{3}{5}$	$h_{33}(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{13}{180}$
$h_{25}(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{1}{36}$	$h_{34}(x) = \frac{1}{180}x^2 + \frac{2}{15}x + 1$
$h_{26}(x) = \frac{1}{180}x^2 + \frac{7}{90}x + \frac{2}{9}$	$h_{35}(x) = \frac{1}{180}x^2 + \frac{1}{45}x - \frac{1}{36}$

F h'_r **Polynomials for** $\{3, 5, 6\}$

$$M' = LCM(GCD(3, 5), GCD(3, 6), GCD(5, 6)) = LCM(1, 3, 1) = 3.$$

$$h_0(x) = \frac{1}{180}x^2 + \frac{2}{15}x$$

$$h_1(x) = \frac{1}{180}x^2 + \frac{1}{45}x$$

$$h_2(x) = \frac{1}{180}x^2 + \frac{7}{90}x$$

$$h_3(x) = \frac{1}{180}x^2 + \frac{2}{15}x$$

The smallest b_i is $-\frac{77}{180} = -0.4277$ and the largest b_i is 1. The difference between the largest and smallest is 1.4277.

G Some of the h_r Polynomials for $\{1, 4, 6, 11\}$

$$M = LCM(1, 4, 6, 11) = 264$$

$$h_{22}(x) = \frac{1}{1584}x^3 + \frac{1}{48}x^2 + \frac{101}{528}x - \frac{9}{176}$$

$$h_{87}(x) = \frac{1}{1584}x^3 + \frac{1}{48}x^2 + \frac{101}{528}x - \frac{9}{176}$$

$$h_{98}(x) = \frac{1}{1584}x^3 + \frac{1}{48}x^2 + \frac{7}{33}x - \frac{23}{396}$$

H h'_r Polynomials for $\{1, 4, 6, 11\}$

$$M' = LCM(GCD(4, 6), GCD(4, 11), GCD(6, 11)) = LCM(2, 1, 1) = 2.$$

$$h'_0(x) = \frac{1}{1584}x^3 + \frac{1}{48}x^2 + \frac{101}{528}x$$

$$h'_1(x) = \frac{1}{1584}x^3 + \frac{1}{48}x^2 + \frac{101}{528}x$$

The smallest b_i is $-\frac{23}{396} = -0.05808$ and the largest b_i is $\frac{15}{11} = 1.3636$. The difference between the largest and smallest is 1.4168.

I Some of the h_r Polynomials for $\{1, 19, 19, 20\}$

$$M = LCM(1, 19, 19, 20) = 380.$$

For all $0 \leq k \leq 19$

$$h_{19k}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 - \frac{1}{228}x + 1$$

$$h_{19k+17}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 - \frac{1}{228}x + \frac{35}{16}$$

$$h_{19k+18}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 - \frac{127}{4332}x + \frac{5279}{7220}$$

J h'_r Polynomials for $\{1, 19, 19, 20\}$

$$M' = LCM(GCD(19, 19), GCD(19, 20)) = LCM(19, 1) = 19.$$

$h'_0(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 - \frac{1}{228}x + 1$	$h'_5(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{341}{4332}x + \frac{3191}{5776}$
$h'_1(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{77}{4332}x + \frac{28307}{28880}$	$h'_6(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{377}{4332}x + \frac{2883}{7220}$
$h'_2(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{161}{4332}x + \frac{6623}{7220}$	$h'_7(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{401}{4332}x + \frac{7047}{28880}$
$h'_3(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{233}{4332}x + \frac{23671}{28880}$	$h'_8(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{413}{4332}x + \frac{9}{95}$
$h'_4(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{293}{4332}x + \frac{251}{361}$	$h'_9(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{413}{4332}x - \frac{233}{5776}$

$h'_{10}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{401}{4332}x - \frac{221}{1444}$	$h'_{15}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{161}{4332}x - \frac{549}{5776}$
$h'_{11}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{377}{4332}x - \frac{6793}{28880}$	$h'_{16}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{77}{4332}x + \frac{177}{1805}$
$h'_{12}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{341}{4332}x - \frac{503}{1805}$	$h'_{17}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 - \frac{1}{228}x + \frac{10707}{28880}$
$h'_{13}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{293}{4332}x - \frac{7949}{28880}$	$h'_{18}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 - \frac{127}{4332}x + \frac{5279}{7220}$
$h'_{14}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 + \frac{233}{4332}x - \frac{313}{1444}$	$h'_{19}(x) = \frac{1}{43320}x^3 + \frac{59}{28880}x^2 - \frac{1}{228}x + \frac{35}{16}$

The smallest b_i is $-\frac{36731}{5776} = -6.3644$ and the largest b_i is $\frac{12807}{1805} = 7.0953$. The difference between the largest and smallest is 13.4597.

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